Bohnenblust-Hille inequalities for Lorentz spaces via interpolation

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Abstract

We prove that the Lorentz sequence space $\ell_{\frac{2m}{m+1},1}$ is, in a precise sense, optimal among all symmetric Banach sequence spaces satisfying a Bohnenblust-Hille type inequality for m-linear forms or m-homogeneous polynomials on $\mathbb{C}^n$. Motivated by this result we develop methods for dealing with subtle Bohnenblust-Hille type inequalities in the setting of Lorentz spaces. Based on an interpolation approach and the Blei-Fournier inequalities involving mixed type spaces, we prove multilinear and polynomial Bohnenblust-Hille type inequalities in Lorentz spaces with subpolynomial and subexponential constants. Improving a remarkable result of Balasubramanian-Calado-Queffélec, we show an application to the theory of Dirichlet series.

1 Introduction and classical results

In their seminal article [8] Bohnenblust and Hille proved that there exists a positive function f on $\mathbb{N}$ such that for every n and every m-homogeneous polynomial on $\mathbb{C}^n$, the ℓ_p-norm with $p = \frac{2m}{m+1}$ of the set of its coefficients is bounded above by the constant $f(m)$ times the supremum norm of the polynomial on the unit polydisc $\mathbb{D}^n$. The initial interest of this result is that $f(m)$ is independent of the dimension n and, moreover, the exponent $\frac{2m}{m+1}$ is optimal. This result was a key point in the celebrated solution by Bohnenblust and Hille of Bohr's absolute convergence problem for Dirichlet series (see, e.g., [8, 9, 12], or [14]).

Recently, more sophisticated results were obtained and successfully applied to verify several long standing conjectures in the convergence theory for Dirichlet series (and intimately related complex analysis in high dimensions). A striking improvement was given in [11] proving that $f(m)$ in fact grows at most exponentially in m, and a recent result from [5] even states that $f(m)$ is subexponential in the sense that for every $\varepsilon > 0$ there is a constant $C(\varepsilon)$ such that $f(m) \leq C(\varepsilon)(1+\varepsilon)^m$ for each $m \in \mathbb{N}$. Estimates of this type proved to be useful in many different areas of analysis (e.g., the modern $\mathcal{H}_p$-theory of Dirichlet series and (the intimately connected) infinite dimensional holomorphy (see, e.g., [4] or [14]), the study of summing polynomials in Banach spaces (see, e.g., [1, 13], or [15]), and even in quantum information theory (see [22]) and more generally in Fourier analysis of Boolean functions. A good general reference in this area is the recent book of O'Donnell [23].

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Our aim is to prove multilinear and polynomial Bohnenblust-Hille inequalities in the setting of Lorentz spaces. In the remaining part of this introduction we give more precise details on the state of art of BH-inequalities (multilinear and polynomial), and isolate the two natural problems we are mainly concerned with.

We will consider Banach sequence spaces $\big(X(I), \|\cdot\|_X\big)$ of $\mathbb{C}$-valued sequences $(x_i)_{i\in I}$ which are defined over arbitrarily given (index) sets I. In what follows Lorentz spaces will play an important role. Given $1 \le p < \infty$ and $1 \le q \le \infty$, the Lorentz space $\ell_{p,q}(I)$ ($\ell_{p,q}$ for short) on a nonempty set I consists of all $x = (x_i)_{i\in I}$ for which the expression

$$\|x\|_{\ell_{p,q}} = \begin{cases} \Big(\sum_{k\in J} x_k^{*q}\big(k^{q/p} - (k-1)^{q/p}\big)^q\Big)^{1/q} & \text{if} \quad q < \infty \\ \sup_{k\in J} k^{1/p} x_k^* & \text{if} \quad q = \infty \end{cases} \tag{1}$$

is finite. Here, as usual, for a given $x = (x_i)_{i\in I} \in \ell_\infty(I)$, we denote by $x^* = (x_j^*)_{j\in J}$ the non-increasing rearrangement of x defined by

$$x_j^* = \inf\big\{\lambda > 0;\, \mathrm{card}\{i \in I;\, |x_i| > \lambda\} \le j\big\}, \quad j \in J,$$

where $J = \{1, ..., n\}$ whenever $\mathrm{card}\, I = n$, and $J = \mathbb{N}$ whenever I is infinite. The expression (1) is a norm if $q \le p$, and a quasi norm if $q > p$. In the second case $\|\cdot\|_{\ell_{p,q}}$ is equivalent to a norm. Of course, $\ell_{p,p}$ is the Minkowski space ℓ_p since the map $x \mapsto x^*$ is an isometry.

The following two finite index sets will be of special interest: For each $m, n \in \mathbb{N}$

$$\mathcal{M}(m,n) = \big\{\mathbf{i} = (i_1, \dots, i_m)\,;\, i_k \in \mathbb{N}\,,\, 1 \le i_k \le n\big\}$$

and

$$\mathcal{J}(m,n) = \big\{\mathbf{j} \in \mathcal{M}(m,n)\,;\, j_1 \le j_2 \le \dots \le j_m\big\}\,.$$

Below we explain the two inequalities we are interested in, the so-called multilinear and polynomial Bohnenblust-Hille inequalities, and we motivate the two problems we intend to handle.

The multilinear BH-inequality. Given a Banach sequence space X (defined over arbitrary index sets) and $m \in \mathbb{N}$, we denote by

$$\mathrm{BH}_X^{\mathrm{mult}}(m) \in \big[1, \infty\big]$$

the best constant $C \ge 1$ such that for every n and every complex matrix $a = (a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$ we have

$$\big\|(a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}\big\|_X \le C\|a\|_\infty\,, \tag{2}$$

where

$$\|a\|_\infty = \sup_{\substack{\|(x_i^k)_{i=1}^n\|_\infty \le 1 \\ 1\le k\le m}} \Big|\sum_{\mathbf{i}=(i_1,\dots,i_n)\in\mathcal{M}(m,n)} a_{\mathbf{i}}\, x_{i_1}^1 \dots x_{i_m}^m\Big|\,.$$

For the sake of completeness we give a short review of the history of the inequalities from (2) emphasizing those results, old and very recent ones, which are of relevance for this article. (For more on that we once again refer to [14].) The case $m = 2$ reflects a famous result of Littlewood [21]:

$$\mathrm{BH}_{\ell_{\frac{4}{3}}}^{\mathrm{mult}}(2) < \infty.$$

Solving Bohr's so-called absolute convergence problem on Dirichlet series Bohnenblust and Hille in [8] studied the case of arbitrary m and proved that

$$\mathrm{BH}^{\mathrm{mult}}_{\ell_{\frac{2m}{m+1}}}(m) < \infty. \tag{3}$$

This result was improved by Fournier and Blei [7, 16] showing that even,

$$\mathrm{BH}^{\mathrm{mult}}_{\ell_{\frac{2m}{m+1},1}}(m) < \infty. \tag{4}$$

In Section 4 we give a modified version of their proof from [7].
Finally, it turned out in a recent article [5] by Bayart, Pellegrino, and Seoane that the constants in (3) are subpolynomial in the following sense: There is a constant $\kappa > 1$ such that for all m we have

$$\mathrm{BH}^{\mathrm{mult}}_{\ell_{\frac{2m}{m+1}}}(m) \le \kappa\, m^{\frac{1-\gamma}{2}}, \tag{5}$$

where γ is the Euler-Masceroni constant. Note that there exits a uniform constant $C > 0$ such that for any finite index sets I

$$\big\|\ell_p(I) \hookrightarrow \ell_{p,1}(I)\big\| \le C \log(\operatorname{card} I)\,, \tag{6}$$

hence by (5) there exits $\delta > 1$ such that for each m, n and every matrix $(a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$,

$$\big\|(a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}\big\|_{\frac{2m}{m+1},1} \le m^{\delta}\,(\log n)\|a\|_\infty\,.$$

In view of this, and comparing with (4) and (5), the following natural question appears.

Problem 1. *Does there exist a constant $\delta > 0$ such that for each m we have*

$$\mathrm{BH}^{mult}_{\ell_{\frac{2m}{m+1},1}}(m) \le m^{\delta}\,.$$

We provide far-reaching partial solutions extending all results mentioned before. The main contributions are given in the Theorems 6 and 12.

The polynomial BH-inequality. Every m-homogenous polynomial

$$P(z) = \sum_{\substack{\alpha\in\mathbb{N}_0^n\\ |\alpha|=m}} c_\alpha z^\alpha$$

in n complex variables $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ can be uniquely rewritten in the form

$$P(z) = \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}}\, z_{j_1}\dots z_{j_m}\,, \tag{7}$$

and we denote its supremum norm by

$$\|P\|_\infty = \sup_{\|(z_i)_{i=1}^n\|_\infty\le 1} \Big|\sum_{\mathbf{j}=(i_1,\dots,i_n)\in\mathcal{J}(m,n)} c_{\mathbf{j}}\; z_{j_1}\dots z_{j_m}\Big|.$$

Given a Banach sequence space X (defined over an arbitrary index set) and $m \in \mathbb{N}$, we denote by

$$\mathrm{BH}_X^{\mathrm{pol}}(m) \in \big[1, \infty\big]$$

the best constant $C \geq 1$ such that for every n and every m-homogeneous polynomial P as in (7) we have

$$\big\|(c_{\mathbf{j}}(P))_{\mathbf{j} \in \mathcal{J}(m,n)}\big\|_X \leq C \|P\|_\infty \,. \tag{8}$$

Let us again give a short review of the most important results on such inequalities (for more information see again [14]): Inventing polarization, Bohnenblust and Hille in [8] deduced from (3) that

$$\mathrm{BH}^{\mathrm{pol}}_{\ell_{\frac{2m}{m+1}}}(m) < \infty. \tag{9}$$

The fact that $p = \frac{2m}{m+1}$ is optimal here was a crucial step in the solution of Bohr's so-called absolute convergence problem. Again, mainly motivated through problems on the general theory of Dirichlet series and holomorphic functions in high dimensions, the first qualitative improvement of the constants was done in [11]: For every $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ such that for all m

$$\mathrm{BH}^{\mathrm{pol}}_{\ell_{\frac{2m}{m+1}}}(m) \leq C(\varepsilon)(\sqrt{2} + \varepsilon)^m. \tag{10}$$

Bayart, Pellegrino, and Seoane proved in [5] that these constants even are subexponential in the following sense:

$$\mathrm{BH}^{\mathrm{pol}}_{\ell_{\frac{2m}{m+1}}}(m) \leq C(\varepsilon)(1 + \varepsilon)^m. \tag{11}$$

We are going to see that a standard polarization argument extends (9) to Lorentz spaces:

$$\mathrm{BH}^{\mathrm{pol}}_{\ell_{\frac{2m}{m+1},1}}(m) < \infty \,, \tag{12}$$

but the following problem will turn out to be much more challenging.

Problem 2. *To what extent do* (10) *and* (11) *hold when we replace* $\ell_{\frac{2m}{m+1}}$ *by the Lorentz sequence space* $\ell_{\frac{2m}{m+1},1}$.

Subsequent to the case of (10) our main result is given in Theorem 14.

Why do Lorentz spaces play an essential role within the context of Bohnenblust-Hille inequalities? We prove (see Theorem 1) that among all symmetric Banach sequence spaces X satisfying a multilinear or polynomial Bohnenblust-Hille inequality as in (2) or (8) the sequence space $X = \ell_{\frac{2m}{m+1},1}$ is the smallest one (and in this sense the "best").

2 Preliminaries

Throughout the paper, for a given finite set $\{X_i\}_{i \in I}$ of Banach spaces which are all contained in some linear space $\mathcal{X}$, we denote by $\bigoplus_{i \in I} X_i$ the Banach space of all $x \in \bigcap_{i \in I} X_i$ equipped with the norm

$$\|x\|_{\bigoplus_{i \in I} X_i} = \sum_{i \in I} \|x\|_{X_i}.$$

For each $m \in \mathbb{N}$ we denote by $\mathcal{M}(m)$ and $\mathcal{J}(m)$ the union of all $\mathcal{M}(m,n)$ and $\mathcal{J}(m,n)$, $n \in \mathbb{N}$, respectively. We define an equivalence relation in $\mathcal{M}(m,n)$ in the following way: $\mathbf{i} \sim \mathbf{j}$ if there is a permutation σ of $\{1,\dots,m\}$ such that $(i_1,\dots,j_m) = (j_{\sigma(1)},\dots,j_{\sigma(m)})$, and denote by $[\mathbf{i}]$ the equivalence class of $\mathbf{i} \in \mathcal{M}(m,n)$. The following disjoint partition of $\mathcal{M}(m,n)$ will be very useful:

$$\mathcal{M}(m,n) = \bigcup_{\mathbf{j}\in\mathcal{J}(m,n)} [\mathbf{j}]$$

For $1 \le k \le m$, let $\mathcal{P}_k(m)$ denote the set of all subsets of $\{1,\dots,m\}$ with cardinality k. We denote the complement of $S \in \mathcal{P}_k(m)$ in $\{1,\dots,m\}$ by $\widehat{S}$. If $S \in \mathcal{P}_k(m)$, then $\mathcal{M}(S,n)$ stands for all indices $\mathbf{i}\colon S \to \{1,\dots,n\}$, and in the special case $S = \{1,\dots,k\}$ we clearly have that $\mathcal{M}(k,n) = \mathcal{M}(S,n)$. Finally, for $\mathbf{i} \in \mathcal{M}(S,n)$ and $\mathbf{j} \in \mathcal{M}(\widehat{S},n)$ we define $\mathbf{i} \oplus \mathbf{j} \in \mathcal{M}(m,n)$ through

$$\mathbf{i} \oplus \mathbf{j} = \begin{cases} \mathbf{i} & \text{on } S \\ \mathbf{j} & \text{on } \widehat{S}. \end{cases}$$

Given $m,n,k \in \mathbb{N}$ with $1 \le k < m$ and $1 \le p,q \le \infty$, we define on the space $\mathbb{C}^{\mathcal{M}(m,n)}$ of all matrices $a = (a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$ the norm $\|\cdot\|_{(m,n,k,p,q)}$ by

$$\|a\|_{(m,n,k,p,q)} = \sum_{S\in\mathcal{P}_k(m)} \Bigg(\sum_{\mathbf{i}\in\mathcal{M}(S,n)} \Bigg(\sum_{\mathbf{j}\in\mathcal{M}(\widehat{S},n)} |a_{\mathbf{i}\oplus\mathbf{j}}|^q \Bigg)^{p/q} \Bigg)^{1/p},$$

and denote the corresponding Banach space by

$$\bigoplus_{S\in\mathcal{P}_k(m)} \ell_p(S)\big[\ell_q(\widehat{S})\big].$$

Clearly, this is the ℓ_1-sum of all Banach spaces $\ell_p(S)[\ell_q(\widehat{S})]$, where $\ell_p(S)[\ell_q(\widehat{S})]$ by definition equals $\mathbb{C}^{\mathcal{M}(m,n)}$ normed by

$$\|a\|_{\ell_p(S)[\ell_q(\widehat{S})]} = \Bigg(\sum_{\mathbf{i}\in\mathcal{M}(S,n)} \Bigg(\sum_{\mathbf{j}\in\mathcal{M}(\widehat{S},n)} |a_{\mathbf{i}\oplus\mathbf{j}}|^q \Bigg)^{p/q} \Bigg)^{1/p}.$$

We will consider (classes) Banach lattices. Of particular importance are *symmetric* spaces. We recall that a Banach lattice E on a measure space (Ω,Σ,μ) is said to be symmetric whenever $g \in E$ and $\|f\|_E = \|g\|_E$ provided that $\mu_f = \mu_g$ and $f \in E$. Here μ_f denotes the distribution function of f defined by $\mu_f(\lambda) = \mu\{t \in \Omega;\, |f(t)| > \lambda\}$, $\lambda \ge 0$. Throughout the paper, by a Banach sequence lattice on a finite or countable set I we mean a real or complex Banach lattice E on the measure space $(I, 2^I, \mu)$ (for short on I), where μ is counting measure. In the case when E is symmetric, E is said to be a symmetric Banach (sequence) space.

A symmetric space E is called *fully symmetric* whenever it is an exact interpolation space between $L_1(\mu)$ and $L_\infty(\mu)$, i.e., for any linear operator $T\colon L_1(\mu)+L_\infty(\mu) \to L_1(\mu)+L_\infty(\mu)$ such that $\|T\|_{L_1(\mu)\to L_1(\mu)} \le 1$ and $\|T\|_{L_\infty(\mu)\to L_\infty}(\mu) \le 1$, we have that T maps E into E and $\|T\|_{E\to E} \le 1$. It is well known that symmetric spaces that have the Fatou property or have order continuous norm are fully symmetric (see, e.g., [6, 20]).

We will need the concept of discretization of a Banach lattice. Let (Ω,Σ,μ) be a measure space and let $d = \{\Omega_k\}_{k=1}^N \subset \Sigma$ be a measurable partition of Ω, i.e., $\Omega = \bigcup_{k=1}^N \Omega_k$ where

$\Omega_i \cap \Omega_j = \emptyset$ for each $i, j \in \{1, \dots, N\}$ with $i \neq j$. Then, given a Banach lattice X on (Ω, Σ, μ), the discretization X^d is the Banach space of all simple function $f \in X$ of the form $f = \sum_{k=1}^N \xi_k \chi_{\Omega_k} \in X$ equipped with the induced norm from X.

The notion of Lorentz spaces over arbitrary measure spaces will be essential in what follows. Given a measure space (Ω, Σ, μ) and $0 < p < \infty$, $0 < q \leq \infty$, the Lorentz space $L_{p,q}(\Omega, \mu)$ ($L_{p,q}(\Omega)$ or $L_{p,q}$ for short) is defined to be the space of all (equivalence classes of) measurable functions f on Ω equipped with the quasi norm

$$\|f\|_{L_{p,q}} = \begin{cases} \left(\frac{q}{p}\int_0^\infty f^*(t)^q t^{\frac{q}{p}-1}\, dt\right)^{1/q} & \text{if} \quad q < \infty \\ \sup_{t>0} t^{1/p} f^*(t) & \text{if} \quad q = \infty\,, \end{cases}$$

where f^* is the decreasing rearrangement of f defined on $[0, \infty)$ by

$$f^*(t) = \inf\big\{s > 0;\ \mu_f(s) \leq t\big\}$$

(We adopt the convention $\inf \emptyset = \infty$.) In the case when $\Omega = I$ is a non-empty set and μ is its counting measure, the space $L_{p,q}(\Omega, \mu)$ in fact coincides with the Lorentz sequence space $\ell_{p,q}(I)$ already defined in (1). Indeed, in this case, given a function $f = x$ on $\Omega = I$ we have $x_k^* = f^*(t)$ for every $t \in [k-1, k)$, $k \in J$, where $J = \{1, ..., \operatorname{card} I\}$ if I is finite, and $J = \mathbb{N}$ if I is infinite. Thus $\|f\|_{L_{p,q}} = \|x\|_{\ell_{p,q}}$, where the latter norm is as defined by the formula (1).

We recall that the Köthe dual space $(\ell_{p,1})'$ of the Lorentz space $\ell_{p,1} = \ell_{p,1}(I)$ coincides with the Marcinkiewicz space m_p which consists of all complex sequences $x = (x_i)_{i \in I}$ such that

$$\|x\|_{m_p} = \sup_{k \in J} \frac{\sum_{j=1}^k x_j^*}{k^{1/p}} < \infty\,,$$

and which with this norm forms a Banach space. Moreover, we note that by standard comparison with the integral of t^α on $[1, N]$, we for each $N \in \mathbb{N}$ and every $\alpha \in (0,1)$ have

$$\sum_{k=1}^N \frac{1}{k^\alpha} < \frac{1}{1-\alpha} N^{1-\alpha}\,. \tag{13}$$

Combining this inequality (for $\alpha = 1/p$) with $x_k^* \leq k^{-1/p}\|x\|_{\ell_{p,\infty}}, k \in J$ yields

$$m_p = \ell_{p,\infty}$$

up to equivalent norms:

$$\frac{1}{p'}\,\|x\|_{m_p} \leq \|x\|_{\ell_{p,\infty}} \leq \|x\|_{m_p}, \quad x \in \ell_{p,\infty}\,.$$

(As usual we write $1/p' := 1 - 1/p$.) Many of our arguments will be based on interpolation theory. Here we recall some of its basic concepts and provide some special facts we are going to use. Recall that if $\vec{A} = (A_0, A_1)$ is a quasi normed couple, then for any $a \in A_0 + A_1$ we define the K-functional

$$K(t, a; \vec{A}) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1};\ a_0 + a_1 = a\}, \quad t > 0.$$

For $0 < \theta < 1$, $0 < q < \infty$, the real interpolation space $(A_0, A_1)_{\theta,q}$ is the space of all $a \in A_0 + A_1$ equipped with the quasi norm

$$\|a\|_{\theta,q} = \Big(\int_0^\infty \big(t^{-\theta} K(t, a; \vec{A})\big)^q \frac{dt}{t}\Big)^{1/q}\,,$$

with an obvious modification for $q = \infty$.

The following well known and easily verified interpolation property holds: If (A_0, A_1) and (B_0, B_1) are two quasi normed couples and $T\colon (A_0, A_1) \to (B_0, B_1)$ and $T\colon A_0 + A_1 \to B_0 + B_1$ is such that both restrictions $T\colon A_j \to B_j$ are bounded with the quasi norms M_j, then $T\colon (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$ is also bounded, and for its quasi norm M we have

$$M \le M_0^{1-\theta} M_1^{\theta}.$$

Lorentz spaces arise naturally in the real interpolation method since most of their important properties can be derived from real interpolation theorems. We briefly review some basic definitions. The couple (L_1, L_∞) is especially important for the understanding of the space $L_{p,q}$. It is well known that for every $f \in L_1 + L_\infty$,

$$K(t, f; L_1, L_\infty) = \int_0^t f^*(s)\, ds = t f^{**}(t), \qquad t > 0.$$

Hence, for each $\theta \in (0,1)$

$$\|f\|_{\theta,q} = \Big(\int_0^\infty [t^{1-\theta} f^{**}(t)]^q \frac{dt}{t} \Big)^{1/q}.$$

An immediate consequence of Hardy's inequality is the following well known formula, which states that for $1 < p < \infty$, $1 \le q \le \infty$ and $\theta = 1 - 1/p$

$$(L_1, L_\infty)_{\theta,q} = L_{p,q}\,,$$

and moreover

$$\frac{1}{p'} \|f\|_{(L_1,L_\infty)_{\theta,q}} \le \|f\|_{L_{p,q}} \le \|f\|_{(L_1,L_\infty)_{\theta,p}}\,.$$

Moreover, the following result will be used (which follows from the more general result stated in [18, Theorem 4.3]): Let $1/p = (1-\theta)/p_0 + \theta/p_1$, $0 < p_0$, $p_1 < \infty$, $p_0 \ne p_1$ and $0 < q \le \infty$. Then, up to equivalent norms, we have

$$(L_{p_0}, L_{p_1})_{\theta,q} = L_{p,q}.$$

More precisely,

$$\begin{aligned} C^{-1} \theta^{-\min(1/q,1/p_0)} (1-\theta)^{-\min(1/q,1/p_1)} \left(\frac{p}{q}\right)^{1/q} \|f\|_{L_{p,q}} & \\ \le \|f\|_{(L_{p_0},L_{p_1})_{\theta,q}} & \qquad (14) \\ \le C \theta^{-\max(1/q,1/p_0)} (1-\theta)^{-\max(1/q,1/p_1)} \left(\frac{p}{q}\right)^{1/q} \|f\|_{L_{p,q}}\,, & \end{aligned}$$

where $C > 0$ is a universal constant.

We will also make intensive use of complex interpolation, and denote by $[A_0, A_1]_\theta$ the complex interpolation spaces as defined for example in [10]. We recall that if X_0 and X_1 are two complex Banach lattices on a measure space (Ω, Σ, μ), then

$$[X_0, X_1]_\theta = X_0^{1-\theta} X_1^{\theta} \qquad (15)$$

with equality of norms provided one of the spaces has order continuous norm; here following Calderón [10] we denote by $X_0^{1-\theta} X_1^{\theta}$ the Calderón space of all $x \in L^0(\mu)$ such that $|x| \le \lambda |x_0|^{1-\theta} |x_1|^{\theta}$ μ-a.e. on Ω for some constant $\lambda > 0$ and some $x_i \in X_i$ with $\|x_i\|_{X_i} \le 1$ for $i = 0, 1$. We put

$$\|x\|_{X_0^{1-\theta} X_1^{\theta}} = \inf \lambda.$$

3 The optimality of Lorentz spaces

The following theorem motivates our study; we show that in the context of multilinear and polynomial Bohnenblust-Hille inequalities Lorentz spaces are in a certain sense optimal. Before we state and prove these results we recall that if X is a symmetric Banach sequence space on I and χ_A denotes the indicator function of a set $A \subset I$, clearly $\|\chi_A\|_X$ depends only on $\operatorname{card}(A)$. The function $\phi_X(k) = \|\chi_A\|_X$, where $A \subset I$ with $\operatorname{card}(A) = k$, is called the *fundamental function* of X. It is well known (see, e.g., [20, Theorem 2.5.2]) that if $1 \leq p < \infty$ and X is a symmetric Banach sequence space on I such that $\|\chi_A\|_X = \operatorname{card}(A)^{1/p}$ for every indicator function χ_A (i.e., $\phi_X(k) = k^{1/p}$ for every $A \subset I$ with $\operatorname{card}(A) = k$), then $\ell_{p,1} \hookrightarrow X$ with

$$\|x\|_X \leq \|x\|_{\ell_{p,1}}, \quad x \in \ell_{p,1}\,.$$

Thus $\ell_{p,1}$ is the smallest symmetric Banach sequence space on I whose norm coincides with the ℓ_p-norm on indicator functions.

Theorem 1. *Fix a positive integer m. The Lorentz space $\ell_{\frac{2m}{m+1},1}$ is the smallest symmetric Banach sequence space X such that $\mathrm{BH}_X^{mult}(m) < \infty$. Also, the Lorentz space $\ell_{\frac{2m}{m+1},1}$ is the smallest symmetric Banach sequence space X such that $\mathrm{BH}_X^{pol}(m) < \infty$.*

Proof. We follow an argument inspired by [8]. Assume that X is a symmetric Banach sequence space such that $\mathrm{BH}_X^{\mathrm{mult}}(m) < \infty$, i.e., for each $n \in \mathbb{N}$ and every complex matrix $a = (a_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m,n)}$ we have

$$\|a\|_X \leq \ \mathrm{BH}_X^{\mathrm{mult}}(m)\, \|a\|_\infty. \tag{16}$$

It suffices to show that the fundamental function

$$\phi(n) := \Big\| \sum_{i=1}^n e_i \Big\|_X, \quad n \in \mathbb{N}\,, \tag{17}$$

satisfies

$$\phi(n) \leq C(m)\, n^{\frac{m+1}{2m}} \tag{18}$$

for each $n \in \mathbb{N}$. For fixed N choose some $N \times N$ matrix (a_{rs}) such for every r, s we have $|a_{rs}| = 1$ and $\sum_{k=1}^N a_{rk}\overline{a}_{sk} = N\delta_{rs}$ (e.g. $a_{rs} = e^{2\pi i rs/N}$, $1 \leq r, s \leq N$), and define the matrix $a = (a_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m,n)}$ by

$$a_{i_1 \ldots i_m} = a_{i_1 i_2} \cdots a_{i_{m-1} i_m}\,.$$

Since $|a_{i_1 \ldots i_m}| = 1$, we have $\phi(N^m) = \|a\|_X$. We now estimate the norm $\|a\|_\infty$. We do first the trilinear case $m = 3$, where the argument becomes more transparent. We take $x, y, z \in \mathbb{C}^N$ with supremum norm ≤ 1, then, using the Cauchy-Schwarz inequality and the properties of the

matrix, we have

$$\begin{aligned}
&\Big|\sum_{i,j,k} a_{ij}a_{jk}x_iy_jz_k\Big| \le \sum_k \Big|\sum_{i,j} a_{ij}a_{jk}x_iy_j\Big|\,|z_k| \\
&\le N^{1/2}\bigg(\sum_k \Big|\sum_{i,j} a_{ij}a_{jk}x_iy_j\Big|^2\bigg)^{1/2} = N^{1/2}\bigg(\sum_{\substack{i_1,i_2\\ j_1,j_2}} a_{i_1j_1}\overline{a}_{i_2j_2}x_{i_1}\overline{x}_{i_2}y_{j_1}\overline{y}_{j_2}\sum_k a_{j_1k}\overline{a}_{j_2k}\bigg)^{1/2} \\
&= N^{1/2}N^{1/2}\bigg(\sum_{\substack{i_1,i_2\\ j}} a_{i_1j}\overline{a}_{i_2j}x_{i_1}\overline{x}_{i_2}y_j\overline{y}_j\bigg)^{1/2} = N\bigg(\sum_j \Big|\sum_i a_{ij}x_i\Big|^2|y_j|^2\bigg)^{1/2} \\
&\le N\bigg(\sum_{i_1i_2}\sum_j a_{i_1j}\overline{a}_{i_2j}x_{i_1}\overline{x}_{i_2}\bigg)^{1/2} = N^{3/2}\bigg(\sum_i |x_i|^2\bigg)^{1/2} \le N^{4/2}\,.
\end{aligned}$$

In the general case we take $z^{(1)},\dots,z^{(m)} \in \mathbb{C}^N$, each with supremum norm ≤ 1, and repeat this procedure to get

$$\Big|\sum_{i_1,\dots,i_m=1}^{N} a_{i_1i_2}\cdots a_{i_{m-1}i_m}z^{(1)}_{i_1}\cdots z^{(m)}_{i_m}\Big| \le N^{m/2}\bigg(\sum_{i_1}|z^{(1)}_{i_1}|^2\bigg)^{1/2} \le N^{m/2}N^{1/2}\,. \tag{19}$$

Hence $\|a\|_\infty \le N^{\frac{m+1}{2}}$ for each N, and by (16) we have $\phi(N^m) \le BH_X^{\mathrm{mult}}(m)\,(N^m)^{\frac{m+1}{2m}}$. Since for each positive integer n there is N such that $N^m \le n < (N+1)^m$, we finally obtain (18).

To prove the second statement, we assume that X is a symmetric Banach sequence space such that for each n and every m-homogeneous polynomial $P(z) = \sum_{\substack{\alpha\in\mathbb{N}_0^n\\ |\alpha|=m}} c_\alpha z^\alpha$ we have

$$\Big\|(c_\alpha)_{\substack{\alpha\in\mathbb{N}_0^n\\ |\alpha|=m}}\Big\|_X \le BH_X^{\mathrm{pol}}(m)\|P\|_\infty\,.$$

Following non-trivial ideas of Bohnenblust and Hille from [8] it is possible to modify the proof of the first statement which leads to a sort of deterministic proof of the second statement. Here we give an alternative, probabilistic argument. As in (17) we consider the fundamental function $\phi(n), n \in \mathbb{N}$ of X. Then by the Kahane-Salem-Zygmund inequality (see, e.g., Kahane's book [19]) there is a constant $C_{\mathrm{KSZ}} \ge 1$ such that for every choice of N there are signs $\varepsilon_\alpha = \pm 1$ for which

$$\sup_{z\in\mathbb{D}^N}\Big|\sum_{\substack{\alpha\in\mathbb{N}_0^N\\ |\alpha|=m}} \varepsilon_\alpha z^\alpha\Big| \le C_{\mathrm{KSZ}}\,\bigg(N\binom{m+N-1}{m}\log m\bigg)^{1/2}\,.$$

Since the sequence $\big(\phi(N)/N\big)$ is nonincreasing, and for each N we have

$$\frac{N^m}{m!} \le \binom{N+m-1}{m} \le N^m\,,$$

it follows that $\phi(N^m) \le m!\,\phi\big(\binom{N+m-1}{m}\big)$ for each N. Combining the above estimates we conclude that for each N

$$\phi(N^m) \le BH_X^{pol}(m)\,C_{\mathrm{KSZ}}\,m!\,\sqrt{\log m}\,(N^m)^{\frac{m+1}{2m}}\,.$$

This easily implies that there exists a constant $C(m) > 0$ such that

$$\phi(n) \le C(m)n^{\frac{m+1}{2m}}\,, \quad n \in \mathbb{N}$$

and the conclusion again follows. $\square$

4 Multilinear BH-inequalities for Lorentz spaces revisited

In this section we present a slightly modified proof of (4) which was first given in the paper by Blei and Fournier [7]. We need to prove four preliminary lemmas.

Lemma 2. *For each matrix $a = (a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$ and each $S \subset \mathcal{M}(m,n)$*

$$\frac{\sum_{\mathbf{i}\in S}|a_{\mathbf{i}}|}{E(S)} \le m\,\|a\|_{\ell_{\frac{m}{m-1},\infty}}\,,$$

where

$$E(S) := \max_{1\le k\le m} \operatorname{card}\{i_k;\, \mathbf{i}\in S\}\,.$$

Proof. Clearly

$$k^{\frac{m-1}{m}} a_k^* \le \|a\|_{\ell_{\frac{m}{m-1},\infty}}, \quad 1\le k\le n^m\,.$$

Now note that $\sum_{\mathbf{i}\in S}|a_{\mathbf{i}}|$ has not more that $E(S)^m$ summands, and that $\sum_{k=1}^{E(S)^m} a^*(k)$ sums the first $E(S)^m$ many largest $|a_{\mathbf{i}}|, \mathbf{i}\in S$. As a consequence we obtain by (13) (with $\alpha = 1-1/m$)

$$\sum_{\mathbf{i}\in S}|a_{\mathbf{i}}| \le \sum_{k=1}^{E(S)^m} a_k^* \le \|a\|_{\ell_{\frac{m}{m-1},\infty}} \sum_{k=1}^{E(S)^m} k^{-\frac{m-1}{m}} \le m\,\|a\|_{\ell_{\frac{m}{m-1},\infty}} E(S)\,,$$

as desired. $\square$

Lemma 3. *For each matrix $a = (a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$ the index set $\mathcal{M}(m,n)$ splits into a union of m subsets S_k such that for every $1\le q<\infty$,*

$$\max_{1\le k\le m} \left\|a^{S_k}\right\|_{\ell_\infty(\{k\})\left[\ell_q(\widehat{\{k\}})\right]} \le m^{1/q}\,\|a\|_{\ell_{\frac{qm}{m-1},\infty}}\,,$$

where for $S\subset\mathcal{M}(m,n)$ we put $a^S = a_{\mathbf{i}}$ for $\mathbf{i}\in S$ and $a^S = 0$ for $\mathbf{i}\in S$.

Proof. Note first that it suffices to show the desired inequality for $q=1$; for arbitrary $1<q<\infty$ apply the case $q=1$ to $|a|^{1/q}$ instead of a. In view of Lemma 2 we show that there are appropriate sets S_k for which

$$\max_{1\le k\le m} \left\|a^{S_k}\right\|_{\ell_\infty(\{k\})\left[\ell_1(\widehat{\{k\}})\right]} \le \sup_{S\subset\mathcal{M}(m,n)} \frac{\sum_{\mathbf{i}\in S}|a_{\mathbf{i}}|}{E(S)}\,,$$

and without loss of generality we may assume that the supremum on the right side is ≤ 1. Given $1\le k\le m$, observe that

$$\sum_{\ell=1}^{n} \sum_{\substack{\mathbf{i}\in\mathcal{M}(m,n)\\ i_k=\ell}} |a_{\mathbf{i}}| \;\le\; \sum_{\mathbf{i}\in\mathcal{M}(m,n)} |a_{\mathbf{i}}| \;\le\; E(\mathcal{M}(m,n)) = n.$$

Hence there is some $1\le \ell(k)\le n$ such that for

$$T_k^1 = \left\{\mathbf{j}\in\mathcal{M}(m,n);\, j_k = \ell(k)\right\}$$

we have

$$\sum_{\mathbf{i}\in T_k^1} |a_{\mathbf{i}}| \le 1.$$

Then for

$$N_1 = \mathcal{M}(m,n) \setminus \bigcup_{k=1}^{m} T_k^1$$

we obviously get $E(N_1) \le n-1$. If we now repeat this procedure with N_1 instead of $\mathcal{M}(m,n)$, then we obtain m many new index sets T_k^2, $1 \le k \le m$ in N_1 for which

$$\sum_{\mathbf{i}\in T_k^2} |a_{\mathbf{i}}| \le 1$$

and

$$E(N_2) \le n-2 \quad \text{with} \quad N_2 = \left(\mathcal{M}(m,n) \setminus \bigcup_{k=1}^{m} T_k^1\right) \setminus \left(\bigcup_{k=1}^{m} T_k^2\right).$$

Continuing for $j \in \{3,\dots,n\}$, we find the index sets T_k^j, $1 \le j \le n$, $1 \le k \le m$ such that

$$\sum_{\mathbf{i}\in T_k^j} |a_{\mathbf{i}}| \le 1, \quad 1 \le k \le m,\ 1 \le j \le n \tag{20}$$

and

$$E(N_n) = 0 \quad \text{with} \quad N_n = \mathcal{M}(m,n) \setminus \bigcup_{j=1}^{n} \bigcup_{k=1}^{m} T_k^j\,.$$

Define for $1 \le k \le m$

$$S_k = \bigcup_{j=1}^{n} T_k^j\,.$$

Obviously, we have that $N_n = \emptyset$, and hence

$$\mathcal{M}(m,n) = \bigcup_{k=1}^{m} S_k\,.$$

Finally, for any $1 \le k \le m$

$$\left\|a^{S_k}\right\|_{\ell_\infty(\{k\})\left[\ell_q(\widehat{\{k\}})\right]} = \sup_{1\le j\le n} \sum_{\mathbf{i}\in\mathcal{M}(\widehat{\{k\}},n)} |a^k_{\mathbf{i}\oplus j}| \le \sup_{1\le j\le n} \sum_{\substack{\mathbf{i}\in\mathcal{M}(\widehat{\{k\}},n)\\ \mathbf{i}\oplus j\in\bigcup_{l=1}^{n} T_k^l}} |a_{\mathbf{i}\oplus j}| \le 1\,.$$

Let us comment on the argument for the last estimate: Assume without loss of generality that $n=2$. Then by construction, given $j=1$ or $j=2$, we have that either $\mathbf{i}\oplus j \in T_k^1$ for all $\mathbf{i} \in \mathcal{M}(\widehat{\{k\}},n)$ or $\mathbf{i}\oplus j \in T_k^2$ for all $\mathbf{i} \in \mathcal{M}(\widehat{\{k\}},n)$. The conclusion follows from (20). $\square$

Lemma 4. *For each matrix $a = (a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$ and every $1 \le q < \infty$*

$$\|a\|_{\ell_{\frac{qm}{(q-1)m+1},1}} \le m^{\frac{1}{q}} \sum_{1\le k\le m} \|a\|_{\ell_1(\{k\})\left[\ell_{q'}(\widehat{\{k\}})\right]}\,.$$

Proof. Since for every $1 < r < \infty$ we have $m_r = \ell_{r,\infty}$ with $\|\cdot\|_{\ell_{r,\infty}} \leq \|\cdot\|_{m_r}$ and $(\ell_{r,1})' = m_r$ isometrically, the required inequality follows by Lemma 3 and a simple duality argument: Indeed, take a matrix a and sets S_k according to Lemma 3. Then

$$\begin{aligned}
\sum_{\mathbf{i}\in\mathcal{M}(m,n)} |a_{\mathbf{i}} b_{\mathbf{i}}| &\leq \sum_{1\leq k\leq m} \sum_{\mathbf{i}\in\mathcal{M}(m,n)} |a_{\mathbf{i}} b_{\mathbf{i}}^{S_k}| \\
&\leq \sum_{1\leq k\leq m} \|a\|_{\ell_1(\{k\})\left[\ell_{q'}(\widehat{\{k\}})\right]} \left\|b^{S_k}\right\|_{\ell_\infty(\{k\})\left[\ell_q(\widehat{\{k\}})\right]} \\
&\leq \max_{1\leq k\leq m} \left\|b^{S_k}\right\|_{\ell_\infty(\{k\})\left[\ell_q(\widehat{\{k\}})\right]} \sum_{1\leq k\leq m} \|a\|_{\ell_1(\{k\})\left[\ell_{q'}(\widehat{\{k\}})\right]} \\
&\leq m^{1/q} \|b\|_{\ell_{\frac{qm}{m-1},\infty}} \sum_{1\leq k\leq m} \|a\|_{\ell_1(\{k\})\left[\ell_{q'}(\widehat{\{k\}})\right]} \, ,
\end{aligned}$$

the desired conclusion. ☐

The last lemma needed is the following so-called mixed BH-inequality (this is a simple consequence of the multilinear Khinchine inequality, see e.g., [5, 8], or [12]).

Lemma 5. *For each n and each matrix $a = (a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$ we have*

$$\sum_{j=1}^{n} \Bigg(\sum_{\mathbf{i}\in\mathcal{M}(\widehat{\{k\}},n)} |a_{\mathbf{i}\oplus j}|^2 \Bigg)^{1/2} \leq \sqrt{2}^{m-1} \|a\|_\infty, \quad 1 \leq k \leq m.$$

Combining Lemmas 4 ($q = 2$) and 5 gives the proof of (4). As a by-product we get the following estimate for the constant

$$\mathrm{BH}^{\mathrm{mult}}_{\ell_{\frac{2m}{m+1},1}}(m) \leq m^{1/2}\sqrt{2}^{m-1} \, .$$

We note a disadvantage of this proof, it does not give polynomial growth of $\mathrm{BH}^{\mathrm{mult}}_{\ell_{\frac{2m}{m+1},1}}(m)$ in m as we have for $\mathrm{BH}^{\mathrm{mult}}_{\ell_{\frac{2m}{m+1}}}(m)$ in (5).

4.1 Polynomial growth – part I

We are going to give a first improvement of the result from (5). Our estimate shows that the symmetric Banach sequence space

$$X = \ell_{\frac{2m}{m+1},\frac{2(m-1)}{m}}$$

satisfies the BH-inequality from (2) with a constant growing subpolynomially in m. It is important to note that X is strictly larger than the Lorentz space $\ell_{\frac{2m}{m+1},1}$, however, X has the same fundamental function as $\ell_{\frac{2m}{m+1},1}$ which of course fits with Theorem 1.

Theorem 6. *There exists a constant $\delta > 0$ such that for each m,*

$$\mathrm{BH}^{mult}_{\ell_{\frac{2m}{m+1},\frac{2(m-1)}{m}}}(m) \leq m^{\delta} \, .$$

The proof combines ideas and tools from [7, 8, 21] with some more recent ones from [5]. The following lemma, the proof of which is explicitly included in the proof of [5, Proposition 3.1], is crucial. For $1 \leq p \leq 2$ we write $A_p \geq 1$ for the best constant in the Khinchine-Steinhaus inequality: For each choice of finitely many $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$

$$\|(\alpha_k)_{k=1}^N\|_{\ell_2} \leq A_p \bigg(\int_{\mathbb{T}^N} \Big| \sum_{k=1}^N \alpha_k z_k \Big|^p dz \bigg)^{1/p} ,$$

where dz stands for the normalized Lebesgue measure on the N-dimensional torus $\mathbb{T}^N$. Recall that $A_p \leq \sqrt{2}$ for all $1 \leq p \leq 2$.

Lemma 7. *For each n, each matrix $a = (a_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m,n)}$, and each $1 \leq k < m$ we have*

$$\|a\|_{(m,n,k,\frac{2k}{k+1},2)} \leq A_{\frac{2k}{k+1}}^{m-k} \ \mathrm{BH}_{\ell_{\frac{2k}{k+1}}}^{mult}(k) \ \|a\|_\infty .$$

The second lemma needed is an immediate consequence of [7, Theorem 7.2].

Lemma 8. *For each $1 \leq q < \infty$ there is a constant $C_q \geq 1$ such that for each $1 \leq t < q$ and each matrix $a = (a_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m,n)}$*

$$\|a\|_{\ell_{\frac{mqt}{mq+t-q},t}} \leq C_q \, m \, \|a\|_{(m,n,m-1,t,q)} .$$

Proof of Theorem 6. For $q = 2$ and $t = \frac{2(m-1)}{m}$ we have $\frac{mqt}{mq+t-q} = \frac{2m}{m+1}$. Hence, given a matrix $a = (a_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m,n)}$, Lemma 8 yields

$$\|a\|_{\ell_{\frac{2m}{m+1},\frac{2(m-1)}{m}}} \leq C_2 m \|a\|_{(m,n,m-1,\frac{2(m-1)}{m},2)}.$$

Moreover, by Lemma 7 we have

$$\|a\|_{(m,n,m-1,\frac{2(m-1)}{m},2)} \leq A_{\frac{2(m-1}{m}} \ \mathrm{BH}_{\ell_{\frac{2(m-1)}{m}}}^{\mathrm{mult}}(m-1) \, \|a\|_\infty .$$

Combining with (5) we conclude (because $A_p \leq \sqrt{2}$ for each $1 \leq p \leq 2$) that

$$\|a\|_{\ell_{\frac{2m}{m+1},\frac{2(m-1)}{m}}} \leq C_2 m \sqrt{2} \kappa (m-1)^{\frac{1-\gamma}{2}} \|a\|_\infty ,$$

as required. ☐

4.2 Polynomial growth – part II

In this section we use complex and real interpolation as well as results from Fournier's article [16] to improve Theorem 6 considerably (Theorem 12). The starting point of what we intend to prove is the following result.

Lemma 9. *For each $m, n, k \in \mathbb{N}$ with $1 \leq k \leq m$ we have that*

$$\Big\| \bigoplus_{S \in \mathcal{P}_k(m)} \ell_1(S)\big[\ell_\infty(\widehat{S})\big] \ \hookrightarrow \ \ell_{\frac{m}{k},1}(\mathcal{M}(m,n)) \Big\| \leq \binom{m}{k}^{-1} .$$

Proof. A variant of this result is mentioned without proof in [16, p. 69] (the special case $k=1$ is given in [16, Theorem 4.1]; for the general case analyze the proof of [16, Theorem 4.1] and use in particular [16, Theorem 3.3] instead of [16, Theorem 3.1]) in combination with Cauchy's inequality. $\square$

We will need the following obvious technical result; since we here are interested in precise norm estimates, we prefer to include a proof.

Lemma 10. *Let J be a finite set, and let Y and X_j, $j \in J$ be Banach lattices on a measure space (Ω, Σ, μ). Then $\bigoplus_{j\in J}\big(X_j^{1-\theta}Y^\theta\big) = \Big(\bigoplus_{j\in J} X_j\Big)^{1-\theta} Y^\theta$ for every $\theta \in (0,1)$ with*

$$\Big\| \bigoplus_{j\in J}\big(X_j^{1-\theta}Y^\theta\big) \hookrightarrow \Big(\bigoplus_{j\in J} X_j\Big)^{1-\theta} Y^\theta \Big\| \le \mathrm{card\,J}$$

and

$$\Big\| \Big(\bigoplus_{j\in J} X_j\Big)^{1-\theta} Y^\theta \hookrightarrow \bigoplus_{j\in J}\big(X_j^{1-\theta}Y^\theta\big) \Big\| \le \mathrm{card\,J}.$$

Proof. Let $x \in \bigoplus_{j\in J}\big(X_j^{1-\theta}Y^\theta\big)$ with norm less than 1. Since $\|x\|_{X_j^{1-\theta}Y^\theta} < 1$ for each $j \in J$, there exist $y_j \in Y$, $x_j \in X_j$ with $\|y_j\|_Y \le 1$, $\|x_j\|_{X_j} \le 1$ for each $j \in J$ such that

$$|x| \le |x_j|^{1-\theta} |y_j|^\theta, \quad j \in J.$$

This implies

$$|x| \le \big(\min_{k\in J} |x_k|\big)^{1-\theta} \big(\max_{k\in J} |y_k|\big)^\theta.$$

Clearly, $\big\|\min_{k\in J}|x_k|\big\|_{\bigoplus_{j\in J} X_j} \le \sum_{j\in J}\|x_j\|_{X_j} \le \mathrm{card\,J}$ and $\|\max_{k\in J}|y_k|\|_Y \le \mathrm{card\,J}$ yield

$$x \in \Big(\bigoplus_{j\in J} X_j\Big)^{1-\theta} Y^\theta$$

with

$$\|x\|_{(\bigoplus_{j\in J} X_j)^{1-\theta}Y^\theta} \le \mathrm{card\,J}.$$

This shows the first estimate from our statement. The proof of the second statement is straightforward. $\square$

Now we use real and complex interpolation to deduce, from Lemma 9, the following result.

Lemma 11. *For each $m, n, k \in \mathbb{N}$ with $1 \le k \le m$ we have*

$$\Big\| \bigoplus_{S\in\mathcal{P}_k(m)} \ell_{\frac{2k}{k+1}}(S)\big[\ell_2(\widehat{S})\big] \;\hookrightarrow\; \ell_{\frac{2m}{m+1},\frac{2k}{k+1}}(\mathcal{M}(m,n))\Big\| \le 2\binom{m}{k}^{3/2}.$$

Proof. We claim that the following norm estimate holds:

$$\Big\| \bigoplus_{S\in\mathcal{P}_k(m)} \ell_1(S)[\ell_2(\widehat{S})] \ \hookrightarrow\ \ell_{\frac{2m}{m+k},1}(\mathcal{M})\Big\| \le \sqrt{\binom{m}{k}}, \tag{21}$$

where $\mathcal{M} = \mathcal{M}(m,n)$. Indeed, combining complex interpolation first with Lemma 10 $\big($with norm $\binom{m}{k}\big)$ and then with Lemma 9 $\big($with norm $\binom{m}{k}^{-1/2}\big)$ we obtain

$$\begin{aligned}\bigoplus_{S\in\mathcal{P}_k(m)} \ell_1(S)[\ell_2(\widehat{S})] &= \bigoplus_{S\in\mathcal{P}_k(m)} \ell_1(S)\left[\Big[\ell_1(\widehat{S}),\ell_\infty(\widehat{S})\Big]_{\frac12}\right] = \bigoplus_{S\in\mathcal{P}_k(m)} \Big[\ell_1(S)[\ell_1(\widehat{S})],\ell_1(S)[\ell_\infty(\widehat{S})]\Big]_{\frac12}\\
&= \bigoplus_{S\in\mathcal{P}_k(m)} \Big[\ell_1(\mathcal{M}),\ell_1(S)[\ell_\infty(\widehat{S})]\Big]_{\frac12} \overset{\le\binom{m}{k}}{\hookrightarrow} \Big[\ell_1(\mathcal{M}), \bigoplus_{S\in\mathcal{P}_k(m)} \ell_1(S)[\ell_\infty(\widehat{S})]\Big]_{\frac12}\\
&\overset{\le\binom{m}{k}^{-1/2}}{\hookrightarrow} \big[\ell_1(\mathcal{M}),\ell_{\frac{m}{k},1}(\mathcal{M})\big]_{\frac12} = \ell_{\frac{2m}{m+k},1}(\mathcal{M}).\end{aligned}$$

Observe that here the last formula holds with equality of norms; to see this note that for every $1<p<\infty$ and $0<\theta<1$ we have by (15)

$$E := [\ell_1(\mathcal{M}),\ell_{p,1}(\mathcal{M})]_\theta = \ell_1(\mathcal{M})^{1-\theta}\ell_{p,1}(\mathcal{M})^\theta\,.$$

Taking Köthe duals we obtain $E' = \ell_\infty(\mathcal{M})^{1-\theta}(m_p(\mathcal{M}))^\theta = (m_p)^{\frac1\theta}$ which for $\theta=\frac12$ and $p=\frac{m}{k}$ gives $E' = m_{\frac{2m}{m-k}}(\mathcal{M})$, and by duality

$$E = \ell_{\frac{2m}{m+k},1}(\mathcal{M})\,.$$

This proves the claim from (21). Now for $\theta_k = \frac{k-1}{k}$ we have

$$[\ell_1(S),\ell_2(S)]_{\theta_k} = \ell_{\frac{2k}{k+1}}(S).$$

Hence we deduce from (21) and again Lemma 10 that

$$\begin{aligned}\bigoplus_{S\in\mathcal{P}_k(m)} \ell_{\frac{2k}{k+1}}(S)[\ell_2(\widehat{S})] &= \bigoplus_{S\in\mathcal{P}_k(m)} [\ell_1(S),\ell_2(S)]_{\theta_k}[\ell_2(\widehat{S})] = \bigoplus_{S\in\mathcal{P}_k(m)} \Big[\ell_1(S)[\ell_2(\widehat{S})],\ell_2(S)[\ell_2(\widehat{S})]\Big]_{\theta_k}\\
&= \bigoplus_{S\in\mathcal{P}_k(m)} \Big[\ell_1(S)[\ell_2(\widehat{S})],\ell_2(\mathcal{M})\Big]_{\theta_k} \overset{\le\binom{m}{k}}{\hookrightarrow} \left[\bigoplus_{S\in\mathcal{P}_k(m)} \ell_1(S)[\ell_2(\widehat{S})],\ \ell_2(\mathcal{M})\right]_{\theta_k}\\
&\overset{\le\binom{m}{k}^{\frac{1-\theta_k}{2}}}{\hookrightarrow} \big[\ell_{\frac{2m}{m+k},1}(\mathcal{M}),\ell_2(\mathcal{M})\big]_{\theta_k},\end{aligned}$$

and so the norm of the inclusion map is less or equal than

$$\binom{m}{k}\binom{m}{k}^{\frac{1-\theta_k}{2}} = \binom{m}{k}^{1+\frac{1}{2k}} \le \binom{m}{k}^{3/2}.$$

We now need the following equality:

$$\big[\ell_{\frac{2m}{m+k},1}(\mathcal{M}),\ell_2(\mathcal{M})\big]_{\theta_k} = \ell_{\frac{2m}{m+1},\frac{2k}{k+1}}$$

with

$$\Big\|\big[\ell_{\frac{2m}{m+k},1}(\mathcal{M}),\ell_2(\mathcal{M})\big]_{\theta_k} \hookrightarrow \ell_{\frac{2m}{m+1},\frac{2k}{k+1}}(\mathcal{M})\Big\| \leq 2.$$

In fact, from (15) it follows that for $1 \leq q_j \leq p_j < \infty$ with $j = 0,1$ and for $\theta \in (0,1)$ we have

$$\big[\ell_{p_0,q_0},\ell_{p_1,q_1}\big]_\theta = (\ell_{p_0,q_0})^{1-\theta}(\ell_{p_1,q_1})^\theta.$$

And further on for $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$ it can be shown similarly as in the non-atomic case in [17, Lemma 4.1] that in the atomic case we have

$$(\ell_{p_0,q_0})^{1-\theta}(\ell_{p_1,q_1})^\theta = \ell_{p,q}$$

with

$$\big\|(\ell_{p_0,q_0})^{1-\theta}(\ell_{p_1,q_1})^\theta \hookrightarrow \ell_{p,q}\big\| \leq 2^{1/p}.$$

Thus taking $\theta = \frac{k-1}{k}$, $q_0 = 1$, $p_0 = \frac{2m}{m+k}$ and $p_1 = q_1 = 2$, we obtain the required embedding. Combining all together, we finally arrive at

$$\Big\|\bigoplus_{S\in\mathcal{P}_k(m)} \ell_{\frac{2k}{k+1}}(S)[\ell_2(\widehat{S})] \hookrightarrow \ell_{\frac{2m}{m+1},\frac{2k}{k+1}}\Big\| \leq 2\binom{m}{k}^{3/2},$$

which completes the proof. $\square$

A combination of (5), Lemma 7 and Lemma 11 leads to the following substantial improvement of Theorem 6.

Theorem 12. *For each $m,k \in \mathbb{N}$ with $1 \leq k \leq m$ we have*

$$\mathrm{BH}^{mult}_{\ell_{\frac{2m}{m+1},\frac{2k}{k+1}}}(m) \leq 2\binom{m}{k}^{3/2} A^{m-k}_{\frac{2k}{k+1}}\ \mathrm{BH}^{mult}_{\ell_{\frac{2k}{k+1}}}(k)\,.$$

In particular, for each k there is some $\delta(k) > 0$ such that for every $m > k$

$$\mathrm{BH}^{mult}_{\ell_{\frac{2m}{m+1},\frac{2(m-k)}{m-k+1}}}(m) \leq m^{\delta(k)}.$$

5 The polynomial BH-inequality for Lorentz spaces

Let us start with a standard polarization argument showing how the multilinear BH-inequality in Lorentz spaces from (4) transfers to a polynomial BH-inequality in Lorentz spaces (as already stated in (12)).

Theorem 13. *Given $m \in \mathbb{N}$, there is a constant $C > 0$ such that for every m-homogeneous polynomial $P = \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}} z_{j_1}\ldots z_{j_m}$ in n complex variables we have*

$$\big\|(c_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}\big\|_{\ell_{\frac{2m}{m+1},1}} \leq C\|P\|_\infty\ ;$$

in other terms,

$$\mathrm{BH}^{pol}_{\ell_{\frac{2m}{m+1},1}}(m) < \infty\,.$$

Proof. Take some m-homogeneous polynomial P as above, and let $a = (a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$ be the associated symmetric matrix. Then for every $\mathbf{j} \in \mathcal{J}(m,n)$ we have

$$c_{\mathbf{j}} = \mathrm{card}[\mathbf{j}]\, a_{\mathbf{j}}\,,$$

and by standard polarization

$$\|a\|_\infty \leq \frac{m^m}{m!}\|P\|_\infty\,.$$

Obviously,

$$\left\|\ell_{p,1}(\mathcal{M}(m,n)) \hookrightarrow \ell_{p,1}(\mathcal{J}(m,n)), (b_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)} \mapsto (b_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}\right\| \leq 1\,.$$

Combining all this we obtain

$$\begin{aligned}
\left\|(c_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}\right\|_{\frac{2m}{m+1},1} &= \left\|\left(\mathrm{card}[\mathbf{j}]a_{\mathbf{j}}\right)_{\mathbf{j}\in\mathcal{J}(m,n)}\right\|_{\ell_{\frac{2m}{m+1},1}} \\
&\leq \left\|\left(\mathrm{card}[\mathbf{i}]a_{\mathbf{i}}\right)_{\mathbf{i}\in\mathcal{M}(m,n)}\right\|_{\ell_{\frac{2m}{m+1},1}} \\
&\leq m!\left\|\left(a_{\mathbf{i}}\right)_{\mathbf{i}\in\mathcal{M}(m,n)}\right\|_{\ell_{\frac{2m}{m+1},1}} \\
&\leq m!\,\mathrm{BH}^{\mathrm{mult}}_{\ell_{\frac{2m}{m+1},1}}(m)\|a\|_\infty \leq m^m\,\mathrm{BH}^{\mathrm{mult}}_{\ell_{\frac{2m}{m+1},1}}(m)\|P\|_\infty\,,
\end{aligned}$$

which is the estimate we aimed at. $\square$

5.1 Hypercontractive growth

We now improve the preceding theorem by showing that for $X = \ell_{\frac{2m}{m+1},1}$ the constant $\mathrm{BH}^{\mathrm{pol}}_X(m)$ in fact has hypercontractive growth in m; this extends (10) from Minkowski spaces $\ell_{\frac{2m}{m+1}}$ to Lorentz spaces $\ell_{\frac{2m}{m+1},1}$.

Theorem 14. *For every $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ such that for each m*

$$\mathrm{BH}^{pol}_{\ell_{\frac{2m}{m+1},1}}(m) \leq C(\varepsilon)\left(\sqrt{2}+\varepsilon\right)^m.$$

Our proof needs four preliminary lemmas. The understanding of the following diagonal operator

$$D(m,n)\colon \mathbb{C}^{\mathcal{M}(m,n),s} \;\hookrightarrow\; \mathbb{C}^{\mathcal{J}(m,n)}\,,\;\; (a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)} \mapsto (\mathrm{card}[\mathbf{j}]^{\frac{m+1}{2m}}a_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}$$

will turn out to be crucial; here $\mathbb{C}^{\mathcal{M}(m,n),s}$ stands for all symmetric matrices in $\mathbb{C}^{\mathcal{M}(m,n)}$, i.e., all matrices $(a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$ for which $a_{\mathbf{i}} = a_{\mathbf{j}}$ whenever $\mathbf{j} \in [\mathbf{i}]$. Moreover, denote for $1 < p < \infty$ by $\ell^s_{p,1}(\mathcal{M}(m,n))$ the subspace $\mathbb{C}^{\mathcal{M}(m,n),s}$ of $\ell_{p,1}(\mathcal{M}(m,n))$, and define similarly for $1 \leq p < \infty$ the subspace $\ell^s_p(\mathcal{M}(m,n))$.

In Lemma 16 we will use interpolation in order to establish norm estimates for these diagonal operators in Lorentz sequence spaces. In order to do so, we need another technical lemma on real interpolation.

Lemma 15. *Let X_0, X_1 be fully symmetric spaces on a measure space (Ω, Σ, μ). If X_0^d and X_1^d are discretizations of X_0 and X_1 generated by the same measurable partition of Ω, then for every $\theta \in (0,1)$ and $1 \le q \le \infty$ the inclusion map* $\mathrm{id}\colon (X_0^d, X_1^d)_{\theta,q} \to (X_0, X_1)_{\theta,q}$ *is an isometric isomorphism, i.e.,*

$$\|f\|_{(X_0^d, X_1^d)_{\theta,q}} = \|f\|_{(X_0,X_1)_{\theta,q}}, \quad f \in (X_0^d, X_1^d)_{\theta,q}.$$

Proof. Let $\{\Omega_k\}_{k=1}^N \subset \Sigma$ be a given measurable partition of Ω. Define the linear map

$$P\colon L_1(\mu) + L_\infty(\mu) \to L_1(\mu) + L_\infty(\mu)\,, \ \ f \mapsto \sum_{k=1}^{N} \Big(\frac{1}{\mu(\Omega_k)} \int_{\Omega_k} f\, d\mu\Big) \chi_{\Omega_k}.$$

Since $P\colon (L_1(\mu), L_\infty(\mu)) \to (L_1(\mu), L_\infty(\mu))$ with $\|P\|_{L_1(\mu)\to L_1(\mu)} \le 1$ and $\|P\|_{L_\infty(\mu)\to L_\infty(\mu)} \le 1$, and X_0 and X_1 are fully symmetric, it follows that

$$P\colon (X_0, X_1) \to (X_0^d, X_1^d)$$

with $\|P\|_{X_j \to X_j^d} \le 1$ for $j \in \{0,1\}$. This implies that for every $f \in X_0^d + X_1^d$ we have (by $P(f) = f$)

$$K(t, f; X_0^d, X_1^d) = K(t, Pf; X_0, X_1) \le K(t, f; X_0, X_1), \quad t > 0.$$

Since the opposite inequality is obvious, the required statement follows. $\square$

The following result will be essential in what follows.

Lemma 16. *There is a uniform constant $L > 0$ such that for each m and n*

$$\Big\|D(m,n)\colon \ell^s_{\frac{2m}{m+1},1}(\mathcal{M}(m,n)) \ \hookrightarrow \ \ell_{\frac{2m}{m+1},1}(\mathcal{J}(m,n))\Big\| \le L\, m\,.$$

Proof. The proof is based on interpolation, and the short forms $\mathcal{M} = \mathcal{M}(m,n)$ as well as $\mathcal{J} = \mathcal{J}(m,n)$ will be used. We claim that

$$\big\|D(m,n)\colon \ell_1^s(\mathcal{M}) \to \ell_1(\mathcal{J})\big\| \le 1, \quad \big\|D(m,n)\colon \ell_2^s(\mathcal{M}) \to \ell_2(\mathcal{J})\big\| \le \sqrt{m}. \tag{22}$$

Indeed, for every $a \in \mathbb{C}^{\mathcal{M}(m,n),s}$ we have

$$\begin{aligned}\big\|D(m,n)a\big\|_{\ell_1(\mathcal{J})} &= \sum_{\mathbf{j}\in\mathcal{J}} \mathrm{card}[\mathbf{j}]^{\frac{m+1}{2m}} |a_{\mathbf{j}}| = \sum_{\mathbf{j}\in\mathcal{J}} \mathrm{card}[\mathbf{j}]^{\frac{m+1}{2m}-1} \mathrm{card}[\mathbf{j}] |a_{\mathbf{j}}| \\ &\le \sum_{\mathbf{j}\in\mathcal{J}} \mathrm{card}[\mathbf{j}] |a_{\mathbf{j}}| = \sum_{\mathbf{i}\in\mathcal{M}} |a_{\mathbf{i}}| = \|a\|_{\ell_1^s(\mathcal{M})}\,,\end{aligned}$$

and

$$\begin{aligned}\big\|D(m,n)a\big\|_{\ell_2(\mathcal{J})} &= \Big(\sum_{\mathbf{j}\in\mathcal{J}} \mathrm{card}[\mathbf{j}]^{\frac{m+1}{m}} |a_{\mathbf{j}}|^2\Big)^{1/2} = \Big(\sum_{\mathbf{j}\in\mathcal{J}} \mathrm{card}[\mathbf{j}]^{\frac{m+1}{m}-1} \mathrm{card}[\mathbf{j}] |a_{\mathbf{j}}|^2\Big)^{1/2} \\ &= (m!)^{\frac{1}{2m}} \Big(\sum_{\mathbf{j}\in\mathcal{J}} \mathrm{card}[\mathbf{j}] |a_{\mathbf{j}}|^2\Big)^{1/2} \le \sqrt{m} \Big(\sum_{\mathbf{i}\in\mathcal{M}} |a_{\mathbf{i}}|^2\Big)^{1/2} = \sqrt{m}\ \|a\|_{\ell_2^s(\mathcal{M})}\end{aligned}$$

which proves (22). We now apply the two sided norm estimate from (14). In the special case when $p_0 = q_0 = 1$, $p_1 = q_1 = 2$, $q = 1$, $\theta = \frac{m-1}{m}$, we have $p = \frac{2m}{m+1}$ and in particular $1 \leq (p/q)^{1/q} = \frac{2m}{m+1} < 2$. Then for $I = \mathcal{M}(m,n)$ or $I = \mathcal{J}(m,n)$,

$$(\ell_1(I), \ell_2(I))_{\frac{m-1}{m},1} = \ell_{\frac{2m}{m+1},1}(I)\,,$$

and there is $C > 0$ such that for all $a \in \mathbb{C}^{\mathcal{M}(m,n),s}$,

$$\frac{m^{\frac{3}{2}}}{C(m-1)}\, \|a\|_{\ell_{\frac{2m}{m+1},1}(I)} \leq \|a\|_{(\ell_1(I),\ell_2(I))_{\frac{m-1}{m},1}} \leq \frac{Cm^2}{m-1} \|a\|_{\ell_{\frac{2m}{m+1},1}(I)}\,. \tag{23}$$

It follows by Lemma 15 that

$$\|a\|_{\left(\ell_1^s(\mathcal{M}),\ell_2^s(\mathcal{M})\right)_{\frac{m-1}{m},1}} = \|a\|_{(\ell_1(\mathcal{M}),\ell_2(\mathcal{M}))_{\frac{m-1}{m},1}}, \qquad a \in \mathbb{C}^{\mathcal{M}(m,n),s}. \tag{24}$$

Now we interpolate; we recall that for every operator T between interpolation couples (A_0, A_1) and (B_0, B_1), and every $0 < \theta < 1$ we have

$$\big\|T\colon (A_0,A_1)_{\theta,1} \to (B_0,B_1)_{\theta,1}\big\| \leq \big\|T\colon A_0 \to B_0\big\|^{1-\theta} \big\|T\colon A_1 \to B_1\big\|^{\theta}.$$

In particular,

$$\begin{aligned} &\big\|D(m,n)\colon\ (\ell_1^s(\mathcal{M}),\ell_2^s(\mathcal{M}))_{\frac{m-1}{m},1} \to (\ell_1(\mathcal{J}),\ell_2(\mathcal{J}))_{\frac{m-1}{m},1}\,\big\| \\ &\leq \big\|D(m,n)\colon \ell_1^s(\mathcal{M}) \to \ell_1(\mathcal{J})\big\|^{\frac{1}{m}} \big\|D(m,n)\colon \ell_2^s(\mathcal{M}) \to \ell_2(\mathcal{J})\big\|^{\frac{m-1}{m}}. \end{aligned}$$

As a consequence we obtain that for every $a \in \mathbb{C}^{\mathcal{M}(m,n),s}$,

$$\begin{aligned} &\frac{m^{\frac{3}{2}}}{C(m-1)}\, \big\|D(m,n)a\big\|_{\ell_{\frac{2m}{m+1},1}(\mathcal{M})} \overset{(23)}{\leq} \big\|D(m,n)a\big\|_{(\ell_1(\mathcal{J}),\ell_2(\mathcal{J}))_{\frac{m-1}{m},1}} \\ &\leq \|D(m,n)\colon \ell_1^s(\mathcal{M}) \to \ell_1(\mathcal{J})\|^{\frac{1}{m}}\, \|D(m,n)\colon \ell_2^s(\mathcal{M}) \to \ell_2(\mathcal{J})\|^{\frac{m-1}{m}}\, \|a\|_{\left(\ell_1^s(\mathcal{M}),\ell_2^s(\mathcal{M})\right)_{\frac{m-1}{m},1}} \\ &\overset{(24)}{=} \|D(m,n)\colon \ell_1^s(\mathcal{M}) \to \ell_1(\mathcal{J})\|^{\frac{1}{m}}\, \|D(m,n)\colon \ell_2^s(\mathcal{M}) \to \ell_2(\mathcal{J})\|^{\frac{m-1}{m}}\, \|a\|_{(\ell_1(\mathcal{M}),\ell_2(\mathcal{M}))_{\frac{m-1}{m},1}} \\ &\overset{(23)}{\leq} \|D(m,n)\colon \ell_1^s(\mathcal{M}) \to \ell_1(\mathcal{J})\|^{\frac{1}{m}}\, \|D(m,n)\colon \ell_2^s(\mathcal{M}) \to \ell_2(\mathcal{J})\|^{\frac{m-1}{m}}\, \frac{Cm^2}{m-1} \|a\|_{\ell_{\frac{2m}{m+1},1}(\mathcal{J})}\,. \end{aligned}$$

Combining the above estimates with (22), we conclude that for every $a \in \mathbb{C}^{\mathcal{M}(m,n),s}$,

$$\big\|D(m,n)a\|_{\ell_{\frac{2m}{m+1},1}(\mathcal{M})} \leq C^2\,\sqrt{m}\sqrt{m}^{\frac{m-1}{m}}\, \|a\|_{\ell_{\frac{2m}{m+1},1}(\mathcal{J})} \leq C^2\, m \|a\|_{\ell_{\frac{2m}{m+1},1}(\mathcal{J})},$$

and this completes the proof. □

For $1 \leq p \leq 2$ define $S_p > 0$ to be the best constant $C > 0$ in the Khinchine-Steinhaus inequality for m-homogeneous polynomials (see, e.g., [3] or also [12]): For every m-homogeneous polynomial P on $\mathbb{C}^n$ we have

$$\left(\int_{\mathbb{T}^n} |P(z)|^2 dz\right)^{1/2} \leq S_p^m \left(\int_{\mathbb{T}^n} |P(z)|^p dz\right)^{1/p};$$

we will here only use the fact that $S_1 \leq \sqrt{2}$. In what follows we will need the following lemma (see [12, Lemma 6.6]) (implicitly contained in [5]), however only in the case $k = 1$.

Lemma 17. *Let* $P = \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}} z_{j_1} \dots z_{j_m}$ *be a* m*-homogeneous polynomial in* n *variables, and let* $a = (a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$ *be its associated symmetric matrix. Then for every* $S \in \mathcal{P}_k(m)$, $1 \le k \le m$ *we have*

$$\left(\sum_{\mathbf{i}\in\mathcal{M}(S,n)} \left(\sum_{\mathbf{j}\in\mathcal{M}(\widehat{S},n)} \mathrm{card}\,[\mathbf{j}] \left|a_{\mathbf{i}\oplus\mathbf{j}}\right|^2 \right)^{\frac{1}{2}\,\frac{2k}{k+1}} \right)^{\frac{k+1}{2k}} \le S_{\frac{2k}{k+1}}^{m-k}\, \frac{(m-k)!m^m}{(m-k)^{m-k}m!}\, B_{\ell_{\frac{2k}{k+1}}}^{mult}(k)\, \|P\|_\infty .$$

The fourth lemma is an immediate consequence of [7, Theorem 3.3]; here we will use only the case $q = 2$.

Lemma 18. *Given* $1 \le q < \infty$*, there is a constant* $C_q \ge 1$ *such that for every matrix* $a = (a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$

$$\|a\|_{\ell_{\frac{mq}{m+q-1},1}} \le C_q\, m\, \|a\|_{(m,n,1,1,q)} .$$

We are now ready to give the proof Theorem 14.

Proof of Theorem 14. Assume that P is an m-homogeneous polynomial on $\mathbb{C}^n$ with coefficients $(c_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}$, and denote the coefficients of the associated symmetric m-linear form A by $(a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$. We have the simple fact that for all $\mathbf{i} \in \mathcal{M}(\{1\},n)$ and all $\mathbf{j} \in \mathcal{M}(\widehat{\{1\}},n)$

$$\mathrm{card}[\mathbf{i}\oplus\mathbf{j}] \le m\,\mathrm{card}[\mathbf{j}] .$$

Hence we deduce from Lemma 16, Lemma 18 ($q = 2$) and Lemma 17 ($k = 1$) that for each m and n

$$\begin{aligned}
\left\|(c_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}\right\|_{\frac{2m}{m+1},1} &= \left\|(\mathrm{card}[\mathbf{i}]a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{J}(m,n)}\right\|_{\frac{2m}{m+1},1} \\
&\le L\,m \left\|\left(\mathrm{card}[\mathbf{i}]^{1-\frac{m+1}{2m}} a_{\mathbf{i}}\right)_{\mathbf{i}\in\mathcal{M}(m,n)}\right\|_{\frac{2m}{m+1},1} \\
&\le L\,mC_2m \left\|\left(\mathrm{card}[\mathbf{i}]^{1-\frac{m+1}{2m}} a_{\mathbf{i}}\right)_{\mathbf{i}\in\mathcal{M}(m,n)}\right\|_{(m,n,1,1,2)} \\
&= L\,mC_2m \max_{S\in\mathcal{P}_1(m)} \sum_{\mathbf{i}\in\mathcal{M}(\{1\},n)} \left(\sum_{\mathbf{j}\in\mathcal{M}(\widehat{\{1\}},n)} \left|\mathrm{card}[\mathbf{i}\oplus\mathbf{j}]^{\frac{m-1}{2m}} a_{\mathbf{i}\oplus\mathbf{j}}\right|^2 \right)^{1/2} \\
&\le L\,mC_2m \max_{S\in\mathcal{P}_1(m)} \sum_{\mathbf{i}\in\mathcal{M}(\{1\},n)} \left(\sum_{\mathbf{j}\in\mathcal{M}(\widehat{\{1\}},n)} \left|(m\mathrm{card}[\mathbf{j}])^{\frac{m-1}{2m}} a_{\mathbf{i}\oplus\mathbf{j}}\right|^2 \right)^{1/2} \\
&\le L\,mC_2m m^{\frac{m-1}{2m}} \max_{S\in\mathcal{P}_1(m)} \sum_{\mathbf{i}\in\mathcal{M}(\{1\},n)} \left(\sum_{\mathbf{j}\in\mathcal{M}(\widehat{\{1\}},n)} \mathrm{card}[\mathbf{j}]^{\frac{m-1}{m}} \left|a_{\mathbf{j}}\right|^2 \right)^{1/2} \\
&\le L\,mC_2m m^{\frac{m-1}{2m}} \max_{S\in\mathcal{P}_1(m)} \sum_{\mathbf{i}\in\mathcal{M}(\{1\},n)} \left(\sum_{\mathbf{j}\in\mathcal{M}(\widehat{\{1\}},n)} \mathrm{card}[\mathbf{j}] \left|a_{\mathbf{j}}\right|^2 \right)^{1/2} \\
&\le L\,mC_2m m^{\frac{m-1}{2m}} \sqrt{2}^{m-1} \times \frac{(m-1)!m^m}{(m-1)^{m-1}m!} \times B_{\ell_1}^{\mathrm{mult}}(1) \times \|P\|_\infty .
\end{aligned}$$

This completes the argument. $\square$

5.2 The Balasubramanian-Calado-Queffélec result revisited

In this section we improve a remarkable result by Balasubramanian-Calado-Queffféffelec [2]. By $\mathcal{P}(^m c_0)$ we denote the linear space of all m-homogeneous continuous polynomials on c_0 which together with the supremum norm on the open unit ball in c_0 forms a Banach space. On the subspace c_{00} of all finite sequences in c_0 each such polynomial has a unique monomial series decomposition $P(z) = \sum_{|\alpha|=m} c_\alpha(P) z^\alpha\,,\ z \in c_{00}$ (or, in different notation, $P(z) = \sum_{\mathbf{j}\in\mathcal{J}(m)} c_{\mathbf{j}}\, z_{\mathbf{j}}\,,\ z \in c_{00}$). A Dirichlet series $D = \sum_n a_n n^{-s}$ is said to be m-homogeneous whenever $[a_n \neq 0 \Rightarrow n = \mathfrak{p}^\alpha]$ ($\mathfrak{p}$ the sequence of primes). All m-homogeneous Dirichlet series $D = \sum_n a_n n^{-s}$ which converge on $[\mathrm{Re} > 0]$ and are such that the holomorphic function $D(s) = \sum_{n=1}^\infty a_n \frac{1}{n^s}\,,\ s \in [\mathrm{Re} > 0]$ is bounded, form (together with the supremum norm on $[\mathrm{Re} > 0]$) the Banach space $\mathcal{H}_\infty^m$.

It is remarkable that there is a unique isometric isomorphism

$$\mathfrak{B}\colon \mathcal{P}(^m c_0) \to \mathcal{H}_\infty^m, \quad P = \sum_{|\alpha|=m} c_\alpha(P) z^\alpha \mapsto D = \sum_n a_n n^{-s}$$

such that $c_\alpha = a_n$ whenever $n = \mathfrak{p}^\alpha$. (For more information see [12, 14], or [24].) Then the following theorem is an immediate consequence of this identification and Theorem 13.

Theorem 19. *For every Dirichlet series $\sum_n a_n \frac{1}{n^s} \in \mathcal{H}_\infty^m$ we have $\big(a_n^*\big) \in \ell_{\frac{2m}{m-1},1}$.*

Note that for every sequence $a = (a_n) \in \ell_{\frac{2m}{m+1},1}$ we have

$$\sum_{n=1}^\infty |a_n| \frac{1}{n^{\frac{m-1}{2m}}} \leq \sum_{n=1}^\infty a_n^* \frac{1}{n^{\frac{m-1}{2m}}} \asymp \|a\|_{\ell_{\frac{2m}{m+1},1}} < \infty\,.$$

In [2] it is proved that for every Dirichlet series $\sum_{n=1}^\infty a_n \frac{1}{n^s} \in \mathcal{H}_\infty^m$

$$\sum_{n=1}^\infty |a_n| \frac{(\log n)^{\frac{m-1}{2}}}{n^{\frac{m-1}{2m}}} < \infty\,. \tag{25}$$

In addition it is shown that the exponent in the log-term is optimal. A natural question appears: How is this result related with the estimate from Theorem 19? To see this let $\ell_1(\omega)$ be the weighted ℓ_1-space with the weight $\omega = (\omega_n)$ given by

$$\omega_n = \frac{(\log n)^{\frac{m-1}{2}}}{n^{\frac{m-1}{2m}}}, \quad n \in \mathbb{N}\,. \tag{26}$$

We observe that $\ell_1(\omega)$ is different from $\ell_{\frac{2m}{m+1},1}$: In fact, if we would have $\ell_1(\omega) \subset \ell_{\frac{2m}{m+1},1}$, or equivalently $\ell_1 \subset \ell_{\frac{2m}{m+1},1}(\omega^{-1})$, then by the closed graph theorem

$$\sup_{n\in\mathbb{N}} \|e_n\|_{\ell_{\frac{2m}{m+1},1}(\omega^{-1})} < \infty\,.$$

But since for each $n \in \mathbb{N}$

$$\|e_n\|_{\ell_{\frac{2m}{m+1},1}(\omega^{-1})} = \left\|\frac{e_n}{\omega_n}\right\|_{\ell_{\frac{2m}{m+1},1}} = \frac{n^{\frac{m-1}{2m}}}{(\log n)^{\frac{m-1}{m}}},$$

we get a contradiction. Similarly, if $\ell_{\frac{2m}{m+1},1} \subset \ell_1(\omega)$, then there would exist a constant $C > 0$ such that for each $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} \frac{(\log n)^{\frac{m-1}{2}}}{n^{\frac{m-1}{2m}}} = \Big\| \sum_{n=1}^{N} e_n \Big\|_{\ell_1(\omega)} \leq C \Big\| \sum_{n=1}^{N} e_n \Big\|_{\ell_{\frac{2m}{m+1},1}} = C\, N^{\frac{m-1}{2m}}\,,$$

which is again impossible. We conclude the paper with the following formal improvement of Theorem 19 and the Balasubramanian-Calado-Queffélec result (25):

Corollary 20. *For each $m \in \mathbb{N}$ and every Dirichlet series $\sum_{n=1}^{\infty} a_n \frac{1}{n^s} \in \mathcal{H}_\infty^m$,*

$$(a_n)_n \in \ell_1(\omega) \cap \ell_{\frac{2m}{m+1},1}\,,$$

where the weight ω is given by (26).

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